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2008 J. Phys. A: Math. Theor. 41 304021

(http://iopscience.iop.org/1751-8121/41/30/304021)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **41** (2008) 304021 (12pp) doi:10.1088/1751-8113/41/30/304021

Coherent states for the quantum mechanics on a compact manifold

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Received 30 November 2007, in final form 28 February 2008
Published 15 July 2008
Online at stacks.iop.org/JPhysA/41/304021

Abstract
We review our recent observations concerning quantum mechanics on such compact manifolds as a circle, torus and sphere.

PACS numbers: 02.20.Sv, 03.65.−w, 03.65.Sq

1. Introduction

In spite of the fact that coherent states are of fundamental importance in quantum physics the theory of these states in the case when the configuration space exhibits nontrivial topology can hardly be called complete. In particular, so far there is no general method for the construction of coherent states for a particle on an arbitrary manifold. As a matter of fact there exists the general Perelomov construction [1] of coherent states for a Lie group G. In the Perelomov approach the coherent states are defined by

$$T(g)|\phi_0\rangle \tag{1.1}$$

where $T(g)$, $g \in G$, is a unitary irreducible representation of the Lie group G in the Hilbert space $\mathcal{H}$, and $|\phi_0\rangle \in \mathcal{H}$ is a fixed vector. Let H be the stability subgroup of the vector $|\phi_0\rangle$, i.e. H is the maximal subgroup with the property

$$T(h)|\phi_0\rangle = \mathrm{e}^{\mathrm{i}\varphi}|\phi_0\rangle, \tag{1.2}$$

where $h \in H$. Since $T(g)|\phi_0\rangle$ and $T(gh)|\phi_0\rangle$, where $h \in H$, refer to the same state, therefore the coherent states are parametrized by cosets belonging to G/H. Now let M be a configuration manifold. As with the standard Glauber–Klauder coherent states for the quantum mechanics on a real line, we assume that the coherent states for a particle on M should be labelled by points of the classical phase space, that is the cotangent bundle T^*M. Clearly T^*M need not, in general, be a quotient G/H mentioned above (homogeneous space), and Perelomov method cannot be applied. Even in the case when the classical phase space T^*M is a homogeneous space, there still remains in the Perelomov approach a nontrivial problem of identification of G, H and $|\phi_0\rangle$. For instance, we have no idea how to apply the Perelomov technique for the circular cylinder, an example of the two-dimensional homogeneous space, which is the phase space for a particle on a circle. In this work we review our recent observations concerning the construction and basic properties of coherent states for the quantum mechanics on such compact manifolds as a circle, torus and sphere. Our treatment is based on the identification of the algebra of quantum observables compatible with constraints and the natural parametrization of the classical phase space. The coherent states are then defined as eigenvectors of a non-Hermitian operator built of elements of the algebra. The unitary operator in the polar decomposition of such non-Hermitian operator describes the position of a particle on a manifold and the Hermitian part is related to the momenta. In section 1 we discuss the coherent states for the quantum mechanics on a circle. Section 2 deals with the coherent states for a particle on a torus. Section 3 is devoted to the coherent states for the quantum mechanics on a sphere.

2. Coherent states for the quantum mechanics on a circle

In this section, we collect the basic facts about the coherent states for a particle on a circle S^1. For an excellent review of the quantum mechanics on a circle including the coherent states we refer to a very recent paper [2]. The algebra adequate for the study of the circular motion is the $e(2)$ algebra

$$[J, X_i] = \mathrm{i}\epsilon_{ij} X_j \qquad [X_i, X_j] = 0 \qquad i, j = 1, 2, \tag{2.1}$$

where J is the angular momentum, X_1 and X_2 are the position operators, ϵ_{ij} is the antisymmetric tensor, and we set $\hbar = 1$. Indeed, the algebra (2.1) has the Casimir operator given in a unitary irreducible representation by

$$X_1^2 + X_2^2 = r^2. \tag{2.2}$$

On introducing the unitary operator $U = \mathrm{e}^{\mathrm{i}\hat{\varphi}}$ representing the position of the quantum particle on a circle

$$U = \frac{1}{r}(X_1 + \mathrm{i}X_2) \tag{2.3}$$

we arrive at the following algebra:

$$[J, U] = U \qquad [J, U^\dagger] = -U^\dagger. \tag{2.4}$$

Consider the eigenvalue equation

$$J|j\rangle = j|j\rangle. \tag{2.5}$$

The operators U and $U^\dagger$ act on the vectors $|j\rangle$ as the rising and lowering operators, respectively

$$U|j\rangle = |j+1\rangle \qquad U^\dagger|j\rangle = |j-1\rangle. \tag{2.6}$$

Demanding the time-reversal invariance of the algebra (2.4) we find that j can be only integer and half-integer [3].

We define the coherent states for a particle on a circle by means of the eigenvalue equation [3]

$$Z|z\rangle = z|z\rangle, \tag{2.7}$$

where

$$Z = \mathrm{e}^{-J+1/2}U \tag{2.8}$$

and the complex number

$$z = \mathrm{e}^{-l+\mathrm{i}\alpha} \tag{2.9}$$

parametrizes the circular cylinder $S^1 \times \mathbb{R}$ which is the classical phase space for a particle moving in a circle. Note that the form of the operator Z 'reconstructs' the parametrization (2.9) of the classical phase space, where l is the classical orbital momentum and α the classical angle. The coherent states specified by (2.7) can be alternatively obtained by means of the Zak transform [4]. It follows that

$$|z\rangle = \sum_{j=-\infty}^{\infty} z^{-j}\,\mathrm{e}^{-j^2/2}|j\rangle, \tag{2.10}$$

where substitution $j \to j - 1/2$ in the case with the half-integer j is understood. In view of (2.9) we can write the coherent states in the form

$$|l,\alpha\rangle = \sum_{j=-\infty}^{\infty} \mathrm{e}^{lj-\mathrm{i}j\alpha}\,\mathrm{e}^{-j^2/2}|j\rangle, \tag{2.11}$$

where $|l,\alpha\rangle \equiv |z\rangle$, with $z = \mathrm{e}^{-l+\mathrm{i}\alpha}$. The coherent states are not orthogonal. We have

$$\langle z|w\rangle = \begin{cases} \theta_3\left(\frac{\mathrm{i}}{2\pi}\ln z^*w\middle|\frac{\mathrm{i}}{\pi}\right) & \text{(integer case)} \\ \theta_2\left(\frac{\mathrm{i}}{2\pi}\ln z^*w\middle|\frac{\mathrm{i}}{\pi}\right) & \text{(half-integer case),} \end{cases} \tag{2.12}$$

where integer (half-integer) case refers to j integer (half-integer), and θ_3 and θ_2 are the Jacobi theta-functions [5]. The resolution of the identity for the coherent states can be written as

$$\frac{1}{4\mathrm{i}\pi^{3/2}}\int_{\mathbb{C}} \mathrm{d}z\,\mathrm{d}z^*\frac{\mathrm{e}^{-(\ln|z|)^2}}{|z|^2}|z\rangle\langle z| = I, \tag{2.13}$$

where $\mathbb{C}$ is the complex plane. Using the parametrization (2.9) we can write the (over)completeness condition (2.13) in the form

$$\frac{1}{2\pi^{3/2}}\int_0^{2\pi}\mathrm{d}\alpha\int_{-\infty}^{\infty}\mathrm{d}l\,\mathrm{e}^{-l^2}|l,\alpha\rangle\langle l,\alpha| = I. \tag{2.14}$$

The resolution of the identity (2.13) gives rise to the Bargmann representation such that

$$\langle\phi|\psi\rangle = \frac{1}{4\mathrm{i}\pi^{3/2}}\int_{\mathbb{C}}\mathrm{d}z\,\mathrm{d}z^*\frac{\mathrm{e}^{-(\ln|z|)^2}}{|z|^2}(\phi(z^*))^*\psi(z^*), \tag{2.15}$$

where $\phi(z^*) = \langle z|\phi\rangle$ is the function possessing the Laurent series expansion of the form

$$\phi(z^*) = \begin{cases} \sum_{j=-\infty}^{\infty} c_j z^{*-j} & \text{(integer case)} \\ \sum_{j=-\infty}^{\infty} c_j z^{*-j+1/2} & \text{(half-integer case).} \end{cases} \tag{2.16}$$

The operators act in the representation (2.15) as follows:

$$J\phi(z^*) = -z^*\frac{\mathrm{d}}{\mathrm{d}z^*}\phi(z^*) \qquad U\phi(z^*) = \frac{\phi(ez^*)}{\sqrt{ez^*}} \qquad U^\dagger\phi(z^*) = \frac{z^*}{\sqrt{e}}\phi(\mathrm{e}^{-1}z^*). \tag{2.17}$$

The closeness of the coherent states to points of the classical phase space is illustrated by means of the following relations:

$$\frac{\langle l,\alpha|J|l,\alpha\rangle}{\langle l,\alpha|l,\alpha\rangle} \approx l, \tag{2.18}$$

where a very good approximation of the relative error is [3] $\Delta l/l \approx 2\pi\exp(-\pi^2)\sin(2l\pi)/l$ (for the estimation of the error see also the very recent paper [6]), thus the maximal error arising in the case $l \to 0$ is of order 0.1%. Moreover, we have *exact* equality for l integer or half-integer. Therefore, the parameter l in z can be identified with the classical angular momentum. This interpretation of l is confirmed also by the following approximate relation [3]:

$$\frac{|\langle j|l,\alpha\rangle|^2}{\langle l,\alpha|l,\alpha\rangle} \approx \frac{1}{\sqrt{\pi}}\,\mathrm{e}^{-(j-l)^2} \tag{2.19}$$

which means that the probability distribution of energies for a free particle on a circle is of 'discrete' Gaussian type. The expectation value of the unitary operator $U = \mathrm{e}^{\mathrm{i}\hat{\varphi}}$ representing the position on a circle is given by

$$\frac{\langle l,\alpha|U|l,\alpha\rangle}{\langle l,\alpha|l,\alpha\rangle} \approx \mathrm{e}^{-1/4}\,\mathrm{e}^{\mathrm{i}\alpha}, \tag{2.20}$$

where the approximation is very good. On introducing the relative expectation value

$$\langle\langle U\rangle\rangle_{(l,\alpha)} := \frac{\langle U\rangle_{(l,\alpha)}}{\langle U\rangle_{(0,0)}} \tag{2.21}$$

where $\langle U\rangle_{(l,\alpha)} = \langle l,\alpha|U|l,\alpha\rangle/\langle l,\alpha|l,\alpha\rangle$, we obtain

$$\langle\langle U\rangle\rangle_{(l,\alpha)} \approx \mathrm{e}^{\mathrm{i}\alpha}. \tag{2.22}$$

We conclude that the parameter α can be regarded as a classical angle. We point out that the factor $\mathrm{e}^{-1/4}$ in (2.20) is connected with the fact that U is not diagonal in the coherent state basis. Indeed, it is diagonal in the position representation discussed below (see (2.27)). Finally, the coherent states minimize the following uncertainty relations [7]:

$$\Delta^2(J) + \Delta^2(\hat{\varphi}) \geqslant 1, \tag{2.23}$$

where

$$\Delta^2(J) = \tfrac{1}{4}\ln(\langle\mathrm{e}^{-2J}\rangle\langle\mathrm{e}^{2J}\rangle) \tag{2.24}$$

and

$$\Delta^2(\hat{\varphi}) = \tfrac{1}{4}\ln|\langle U^2\rangle|^{-2}. \tag{2.25}$$

More precisely, in the coherent state we have

$$\Delta^2(J) = 1/2 \qquad \Delta^2(\hat{\varphi}) = 1/2. \tag{2.26}$$

We point out that, in opposition to standard coherent states, the coherent states for the quantum mechanics on a circle are not uniquely determined up to unitary transformation, by the requirement of the saturation of the uncertainty relations (2.23). Namely, they are also minimized by the Schrödinger-cat-like states [8]. For a more detailed discussion of the uncertainty relations for a particle on a circle we also refer besides [7] to the paper [9].

In order to identify wavefunctions corresponding to the coherent states for the quantum mechanics on a circle we now discuss the coordinate representation. This representation is spanned by the eigenvectors $|\varphi\rangle$ of the operator U such that

$$U|\varphi\rangle = \mathrm{e}^{\mathrm{i}\varphi}|\varphi\rangle. \tag{2.27}$$

These vectors form the orthogonal and complete set. The resolution of the identity can be written as

$$\frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\varphi|\varphi\rangle\langle\varphi| = I. \tag{2.28}$$

On taking into account (2.28) we arrive at the realization $L^2(S^1)$ for the abstract Hilbert space of states given by

$$\langle f|g\rangle = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\varphi\, f^*(\varphi)g(\varphi), \tag{2.29}$$

where $f(\varphi) = \langle\varphi|f\rangle$. Because the basis vectors $|j\rangle$ are represented in $L^2(S^1)$ by the functions

$$e_j(\varphi) = \langle\varphi|j\rangle = \mathrm{e}^{\mathrm{i}j\varphi} \tag{2.30}$$

therefore the functions which are the elements of $L^2(S^1)$ are periodic in φ for j integer and antiperiodic in φ for j half-integer. The action of operators J and U in the representation (2.29) is of the following form:

$$Jf(\varphi) = -\mathrm{i}\frac{\mathrm{d}f}{\mathrm{d}\varphi} \qquad Uf(\varphi) = \mathrm{e}^{\mathrm{i}\varphi}f(\varphi). \tag{2.31}$$

Using (2.10) and (2.30) we find that the coherent states are given by [10]

$$f_{(l,\alpha)}(\varphi) = \begin{cases} \theta_3\left(\frac{1}{2\pi}(\varphi-\alpha-\mathrm{i}l)\middle|\frac{\mathrm{i}}{2\pi}\right) & \text{(integer case)}\\ \theta_2\left(\frac{1}{2\pi}(\varphi-\alpha-\mathrm{i}l)\middle|\frac{\mathrm{i}}{2\pi}\right) & \text{(half-integer case),}\end{cases} \tag{2.32}$$

where $f_{(l,\alpha)}(\varphi) = \langle\varphi|l,\alpha\rangle$. The probability density for the coordinates implied by (2.32) is

$$p_{(l,\alpha)}(\varphi) = \frac{|f_{(l,\alpha)}(\varphi)|^2}{\|f_{(l,\alpha)}\|^2} = \begin{cases} \dfrac{\left|\theta_3\left(\frac{1}{2\pi}(\varphi-\alpha-\mathrm{i}l)\middle|\frac{\mathrm{i}}{2\pi}\right)\right|^2}{\theta_3\left(\frac{\mathrm{i}l}{\pi}\middle|\frac{\mathrm{i}}{\pi}\right)} & \text{(integer case)}\\ \dfrac{\left|\theta_2\left(\frac{1}{2\pi}(\varphi-\alpha-\mathrm{i}l)\middle|\frac{\mathrm{i}}{2\pi}\right)\right|^2}{\theta_2\left(\frac{\mathrm{i}l}{\pi}\middle|\frac{\mathrm{i}}{\pi}\right)} & \text{(half-integer case).}\end{cases} \tag{2.33}$$

From computer simulations it follows that the function $p_{(l,\alpha)}(\varphi)$ is peaked at $\varphi=\alpha$. This observation confirms once more the interpretation of the parameter α labelling the coherent states as a classical angle.

We finally remark that the evolution of the wave packets corresponding to coherent states for a free particle on a circle was studied in [10]. In particular, we identified an amazing behaviour of a particle which could be interpreted as quantum jumps on a circle. The quantum deformation of coherent states for the quantum mechanics on a circle was introduced in [11]. The examples of applications of the coherent states for the circular motion include quantum gravity [12], construction of coherent states for a charged particle in a uniform magnetic field [13] and their usage for the study of Husimi phase space distributions and semiclassical propagators [6].

3. Coherent states for a particle on a torus

In this section we summarize the most important facts about the quantum mechanics on a torus. Taking into account the topological equivalence of a two-torus T^2 and the product of two circles as well as the form of the algebra (2.4) we arrive at the following algebra adequate for the study of the motion on a torus:

$$\begin{aligned} &[J_i, U_j] = \delta_{ij}U_j \qquad \left[J_i, U_j^\dagger\right] = -\delta_{ij}U_j^\dagger\\ &[J_i, J_j] = [U_i, U_j] = \left[U_i^\dagger, U_j^\dagger\right] = \left[U_i, U_j^\dagger\right] = 0 \qquad i,j=1,2. \end{aligned} \tag{3.1}$$

Consider the eigenvalue equation

$$\boldsymbol{J}|\boldsymbol{j}\rangle = \boldsymbol{j}|\boldsymbol{j}\rangle \tag{3.2}$$

where $\boldsymbol{J} = (J_1, J_2)$, and $\boldsymbol{j} = (j_1, j_2)$. The operators U_i and $U_j^\dagger$, $i, j = 1, 2$, act on the vectors $|\boldsymbol{j}\rangle$ as the ladder operators, that is

$$U_i|\boldsymbol{j}\rangle = |\boldsymbol{j} + \boldsymbol{e}_i\rangle \qquad U_i^\dagger|\boldsymbol{j}\rangle = |\boldsymbol{j} - \boldsymbol{e}_i\rangle, \tag{3.3}$$

where $\boldsymbol{e}_1 = (1, 0)$, and $\boldsymbol{e}_2 = (0, 1)$ are the unit vectors. Proceeding as with the case of a particle on a circle and demanding the time-reversal invariance we find that there are four possibilities left: j_1-integer and j_2-integer, j_1-integer and j_2-half-integer, j_1-half-integer and j_2-integer, and j_1-half-integer and j_2-half-integer. We designate these cases symbolically by (0, 0), (0, 1/2), (1/2, 0) and (1/2, 1/2), respectively. Motivated by the form of (2.8) we define the coherent states for the quantum mechanics on a torus by means of the eigenvalue equation [14]

$$\boldsymbol{Z}|\boldsymbol{z}\rangle = \boldsymbol{z}|\boldsymbol{z}\rangle, \tag{3.4}$$

where $\boldsymbol{z} = (z_1, z_2) \in \mathbb{C}^2$,

$$Z_i = \mathrm{e}^{-J_i+1/2}U_i \qquad i = 1, 2 \tag{3.5}$$

and

$$z_k = \mathrm{e}^{-l_k+\mathrm{i}\alpha_k} \qquad k = 1, 2 \tag{3.6}$$

so $\boldsymbol{z} \in \mathbb{C}^2$ parametrizes the product of two cylinders $(S^1 \times \mathbb{R}) \times (S^1 \times \mathbb{R})$ which is the classical phase space for a particle on a torus. Some preliminary results concerning coherent states for a torus based on the construction of coherent states for the circle utilizing the Zak transform were described in [4]. The coherent states are given by

$$|\boldsymbol{z}\rangle = \sum_{\boldsymbol{j}\in\mathbb{Z}^2} z_1^{-j_1} z_2^{-j_2}\,\mathrm{e}^{-\boldsymbol{j}^2/2}|\boldsymbol{j}\rangle, \tag{3.7}$$

where $\mathbb{Z}$ is the set of integers and the substitutions $j_2 \to j_2 - 1/2$, $j_1 \to j_1 - 1/2$, and $j_1 \to j_1 - 1/2$ and $j_2 \to j_2 - 1/2$ in the cases (0, 1/2), (1/2, 0) and (1/2, 1/2), respectively, are understood. In the parametrization (3.6) the coherent states take the form

$$|\boldsymbol{l}, \boldsymbol{\alpha}\rangle = \sum_{\boldsymbol{j}\in\mathbb{Z}^2} \mathrm{e}^{\boldsymbol{l}\cdot\boldsymbol{j}-\mathrm{i}\boldsymbol{\alpha}\cdot\boldsymbol{j}}\,\mathrm{e}^{-\boldsymbol{j}^2/2}|\boldsymbol{j}\rangle \tag{3.8}$$

where $|\boldsymbol{l}, \boldsymbol{\alpha}\rangle \equiv |\boldsymbol{z}\rangle$ with $z_k = \mathrm{e}^{-l_k+\mathrm{i}\alpha_k}$, $k = 1, 2$, and $\boldsymbol{u}\cdot\boldsymbol{v} = \sum_{i=1}^2 u_i v_i$,. The overlap of the coherent states is

$$\langle \boldsymbol{z}|\boldsymbol{w}\rangle = \theta_{3-2p}\left(\frac{\mathrm{i}}{2\pi}\ln z_1^* w_1\bigg|\frac{\mathrm{i}}{\pi}\right)\theta_{3-2q}\left(\frac{\mathrm{i}}{2\pi}\ln z_2^* w_2\bigg|\frac{\mathrm{i}}{\pi}\right) \qquad [(p, q) \text{ case}], \tag{3.9}$$

where $p, q = 0, 1/2$. The resolution of the identity in the parametrization (3.6) can be written in the form

$$\frac{1}{4\pi^3}\int_{T^2}\mathrm{d}\boldsymbol{\alpha}\int_{\mathbb{R}^2}\mathrm{d}\boldsymbol{l}\,\mathrm{e}^{-\boldsymbol{l}^2}|\boldsymbol{l}, \boldsymbol{\alpha}\rangle\langle\boldsymbol{l}, \boldsymbol{\alpha}| = I \tag{3.10}$$

where $\int_{T^2}\mathrm{d}\boldsymbol{\alpha} = \int_0^{2\pi}\int_0^{2\pi}\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2$. We omit the discussion of the Bargmann representation which is a straightforward consequence of (3.10) and relations obtained in the previous section in the case of the coherent states for a particle on a circle.

As with the coherent states for the quantum mechanics on a circle our most important criterion to test the correctness of the coherent states for a particle on a torus will be their closeness to the classical phase space in the sense of the formulae on expectation values like (2.18) and (2.22). It follows that

$$\frac{\langle\boldsymbol{l}, \boldsymbol{\alpha}|\boldsymbol{J}|\boldsymbol{l}, \boldsymbol{\alpha}\rangle}{\langle\boldsymbol{l}, \boldsymbol{\alpha}|\boldsymbol{l}, \boldsymbol{\alpha}\rangle} \approx \boldsymbol{l}, \tag{3.11}$$

where the approximation is as good as with the case of the circle, and we have the exact equality for l_k integer and half-integer. Therefore the parameter $\boldsymbol{l}$ can be regarded as a classical angular momentum. The expectation values of the unitary operators U_k, $k = 1, 2$, representing the position of a particle on a circle satisfy

$$\frac{\langle \boldsymbol{l}, \boldsymbol{\alpha}|U_k|\boldsymbol{l}, \boldsymbol{\alpha}\rangle}{\langle \boldsymbol{l}, \boldsymbol{\alpha}|\boldsymbol{l}, \boldsymbol{\alpha}\rangle} \approx \mathrm{e}^{-1/4}\,\mathrm{e}^{\mathrm{i}\alpha_k} \qquad k = 1, 2, \tag{3.12}$$

where the approximation is very good. Using the relative expectation value (see (2.21)) we find

$$\langle\!\langle U_k\rangle\!\rangle_{(\boldsymbol{l},\boldsymbol{\alpha})} \approx \mathrm{e}^{\mathrm{i}\alpha_k} \qquad k = 1, 2. \tag{3.13}$$

It thus appears that α_k can be identified with a classical angle parametrizing the position of a particle on a torus.

We now study the wave packets corresponding to the coherent states for a particle on a torus. Consider the eigenvectors $|\boldsymbol{\varphi}\rangle$ of the operators U_k representing the position of a particle on a torus

$$U_k|\boldsymbol{\varphi}\rangle = \mathrm{e}^{\mathrm{i}\varphi_k}|\boldsymbol{\varphi}\rangle \qquad k = 1, 2. \tag{3.14}$$

The vectors $|\boldsymbol{\varphi}\rangle$ form the orthogonal and complete set. The resolution of the identity can be written in the form

$$\frac{1}{4\pi^2}\int_{T^2} \mathrm{d}\boldsymbol{\varphi}|\boldsymbol{\varphi}\rangle\langle\boldsymbol{\varphi}| = I. \tag{3.15}$$

Hence we obtain the coordinate representation $L^2(T^2)$ for the abstract Hilbert space of states specified by the scalar product

$$\langle f|g\rangle = \frac{1}{4\pi^2}\int_{T^2} \mathrm{d}\boldsymbol{\varphi}\, f^*(\boldsymbol{\varphi})g(\boldsymbol{\varphi}), \tag{3.16}$$

where $f(\boldsymbol{\varphi}) = \langle\boldsymbol{\varphi}|f\rangle$. Taking into account the form of the basis vectors $|\boldsymbol{j}\rangle$ in the coordinate representation

$$e_{\boldsymbol{j}}(\boldsymbol{\varphi}) = \langle\boldsymbol{\varphi}|\boldsymbol{j}\rangle = \mathrm{e}^{\mathrm{i}\boldsymbol{j}\cdot\boldsymbol{\varphi}} \tag{3.17}$$

we find that the functions which are the elements of $L^2(T^2)$ are periodic or antiperiodic ones in φ_1 and φ_2. The operators J_i and U_j, $i, j = 1, 2$ act in the representation (3.11) as follows

$$J_k f(\boldsymbol{\varphi}) = -\mathrm{i}\frac{\partial f}{\partial\varphi_k} \qquad U_k f(\boldsymbol{\varphi}) = \mathrm{e}^{\mathrm{i}\varphi_k} f(\boldsymbol{\varphi}) \qquad k = 1, 2. \tag{3.18}$$

Taking into account (3.8) and (3.12) we get the wavefunctions corresponding to the coherent states of the form

$$f_{(\boldsymbol{l},\boldsymbol{\alpha})}(\boldsymbol{\varphi}) = \theta_{3-2p}\left(\frac{1}{2\pi}(\varphi_1 - \alpha_1 - \mathrm{i}l_1)\middle|\frac{\mathrm{i}}{2\pi}\right)\theta_{3-2q}\left(\frac{1}{2\pi}(\varphi_2 - \alpha_2 - \mathrm{i}l_2)\middle|\frac{\mathrm{i}}{2\pi}\right) \quad [(p, q)\ \text{case}] \tag{3.19}$$

where $f_{(\boldsymbol{l},\boldsymbol{\alpha})}(\boldsymbol{\varphi}) = \langle\boldsymbol{\varphi}|\boldsymbol{l}, \boldsymbol{\alpha}\rangle$, and $p, q = 0, 1/2$. We now restrict for brevity to $(0, 0)$ case that is the case of functions periodic in both φ_1 and φ_2. The probability distribution for the coordinates is then given by

$$p_{(\boldsymbol{l},\boldsymbol{\alpha})}(\boldsymbol{\varphi}) = \frac{|f_{(\boldsymbol{l},\boldsymbol{\alpha})}(\boldsymbol{\varphi})|^2}{\|f_{(\boldsymbol{l},\boldsymbol{\alpha})}\|^2} = \frac{\left|\theta_3\big(\frac{1}{2\pi}(\varphi_1 - \alpha_1 - \mathrm{i}l_1)\big|\frac{\mathrm{i}}{2\pi}\big)\theta_3\big(\frac{1}{2\pi}(\varphi_2 - \alpha_2 - \mathrm{i}l_2)\big|\frac{\mathrm{i}}{2\pi}\big)\right|^2}{\theta_3\big(\frac{\mathrm{i}l_1}{\pi}\big|\frac{\mathrm{i}}{\pi}\big)\theta_3\big(\frac{\mathrm{i}l_2}{\pi}\big|\frac{\mathrm{i}}{\pi}\big)}. \tag{3.20}$$

The computer simulations indicate that the probability distribution (3.20) has a maximum at $\boldsymbol{\varphi} = \boldsymbol{\alpha}$. Such behaviour of the probability density is another evidence in support of the interpretation of the parameter $\boldsymbol{\alpha}$ marking the coherent states as the classical angle.

We finally point out that that the observations obtained in this section for a two-dimensional torus can be immediately generalized to the case of an n-dimensional torus. The possible application of the coherent states for the quantum mechanics on a torus would be the theory of quantum chaos. In fact, the torus is the configuration space for such systems exhibiting deterministic chaos as double pendulum and toroidal pendulum. Furthermore, it seems that the coherent states for the torus would be also of importance in nanotechnology, especially nanoscopic quantum rings [15].

4. Coherent states for the quantum mechanics on a sphere

This section is devoted to the discussion of the basic properties of the coherent states for a particle on a sphere. We first observe that the most natural algebra for the study of the motion on a sphere is the $e(3)$ algebra of the form

$$[J_i, J_j] = \mathrm{i}\varepsilon_{ijk}J_k \qquad [J_i, X_j] = \mathrm{i}\varepsilon_{ijk}X_k \qquad [X_i, X_j] = 0 \qquad i, j, k = 1, 2, 3. \tag{4.1}$$

Indeed, the algebra (4.1) has two Casimir operators given in a unitary irreducible representation by

$$\boldsymbol{X}^2 = r^2 \qquad \boldsymbol{J}\cdot\boldsymbol{X} = \lambda. \tag{4.2}$$

In the following we restrict to the case $\lambda = 0$, then it is plausible to interpret the second equation of (4.1) as the orthogonality condition of the angular momentum $\boldsymbol{J}$ and the radius vector $\boldsymbol{X}$. The basis of the irreducible representation of the $e(3)$ algebra is spanned by the common eigenvectors $\boldsymbol{J}^2$, J_3, $\boldsymbol{X}^2$ and $\boldsymbol{J}\cdot\boldsymbol{X}$ such that

$$\boldsymbol{J}^2|j, m; r\rangle = j(j+1)|j, m; r\rangle \qquad J_3|j, m; r\rangle = m|j, m; r\rangle \tag{4.3a}$$

$$\boldsymbol{X}^2|j, m; r\rangle = r^2|j, m; r\rangle \qquad (\boldsymbol{J}\cdot\boldsymbol{X})|j, m; r\rangle = 0, \tag{4.3b}$$

where j is a nonnegative integer and $-j \leqslant m \leqslant j$. Now the coherent states $|\boldsymbol{z}\rangle$ can be defined as the solution of the eigenvalue equation [16]

$$\boldsymbol{Z}|\boldsymbol{z}\rangle = \boldsymbol{z}|\boldsymbol{z}\rangle, \tag{4.4}$$

where $\boldsymbol{Z}$ is given by

$$\boldsymbol{Z} = \left(\frac{\mathrm{e}^{1/2}}{\sqrt{1+4\boldsymbol{J}^2}}\sinh\frac{1}{2}\sqrt{1+4\boldsymbol{J}^2} + \mathrm{e}^{1/2}\cosh\frac{1}{2}\sqrt{1+4\boldsymbol{J}^2}\right)\frac{\boldsymbol{X}}{r} + \mathrm{i}\left(\frac{2\,\mathrm{e}^{1/2}}{\sqrt{1+4\boldsymbol{J}^2}}\sinh\frac{1}{2}\sqrt{1+4\boldsymbol{J}^2}\right)\boldsymbol{J}\times\frac{\boldsymbol{X}}{r}, \tag{4.5}$$

where the cross designates the vector product. The operator $\boldsymbol{Z}$ and $\boldsymbol{z} \in \mathbb{C}^3$ satisfy

$$\boldsymbol{Z}^2 = 1 \qquad \boldsymbol{z}^2 = 1 \tag{4.6}$$

in accordance with the identification of the classical phase space for a particle on a sphere, i.e. the cotangent bundle T^*S^2 with the affine quadrics [17]. The natural parametrization of $\boldsymbol{z}$ by points of the classical phase space is given by

$$\boldsymbol{z} = \cosh|\boldsymbol{l}|\frac{\boldsymbol{x}}{r} + \mathrm{i}\frac{\sinh|\boldsymbol{l}|}{|\boldsymbol{l}|}\boldsymbol{l}\times\frac{\boldsymbol{x}}{r}, \tag{4.7}$$

where the vectors $\boldsymbol{l}, \boldsymbol{x} \in \mathbb{R}^3$, fulfil $\boldsymbol{x}^2 = r^2$ and $\boldsymbol{l}\cdot\boldsymbol{x} = 0$, that is we assume that $\boldsymbol{l}$ is the classical angular momentum and $\boldsymbol{x}$ is the radius vector of a particle on a sphere. The properties of the operator $\boldsymbol{Z}$, in particular commutativity of its components, are direct consequences of the following matrix representation of this operator:

$$\mathrm{e}^{\boldsymbol{\sigma}\cdot\boldsymbol{J}+1}\boldsymbol{\sigma}\cdot\boldsymbol{X} = \boldsymbol{\sigma}\cdot\boldsymbol{Z}, \tag{4.8}$$

where σ_i, $i = 1, 2, 3$, are the Pauli matrices, which is compatible with the matrix representation of the parametrization (4.7) of the classical phase space such that

$$\mathrm{e}^{\boldsymbol{\sigma}\cdot\boldsymbol{l}}\boldsymbol{\sigma}\cdot\boldsymbol{x} = \boldsymbol{\sigma}\cdot\boldsymbol{z}. \tag{4.9}$$

We now return to (4.4). The coherent states are of the form

$$|\boldsymbol{z}\rangle = \sum_{j=0}^{\infty}\sum_{m=-j}^{j}\mathrm{e}^{-(1/2)j(j+1)}\sqrt{2j+1}\frac{(2|m|)!}{|m|!}\sqrt{\frac{(j-|m|)!}{(j+|m|)!}} \times\left(\frac{-\varepsilon(m)z_1+\mathrm{i}z_2}{2}\right)^{|m|}C_{j-|m|}^{|m|+\frac{1}{2}}(z_3)|j,m;r\rangle, \tag{4.10}$$

where $\varepsilon(m)$ is the sign of m and $C_n^{\alpha}(x)$ are the Gegenbauer polynomials. The coherent states are not orthogonal. We have [18]

$$\langle\boldsymbol{z}|\boldsymbol{w}\rangle = \sum_{j=0}^{\infty}\mathrm{e}^{-j(j+1)}(2j+1)P_j(\boldsymbol{z}^*\cdot\boldsymbol{w}), \tag{4.11}$$

where $P_j(z)$ are the Legendre polynomials. The resolution of the identity can be written as [18]

$$\int_{\boldsymbol{z}^2=1}\mathrm{d}\mu(\boldsymbol{z})|\boldsymbol{z}\rangle\langle\boldsymbol{z}| = I, \tag{4.12}$$

where

$$\mathrm{d}\mu(\boldsymbol{z}) = \frac{1}{4\pi}k_{H^2}(\operatorname{arccosh}(\boldsymbol{z}\cdot\boldsymbol{z}^*), 1)\,\mathrm{d}z_1\,\mathrm{d}z_2\,\mathrm{d}z_3\,\mathrm{d}z_1^*\,\mathrm{d}z_2^*\,\mathrm{d}z_3^* \tag{4.13}$$

and

$$k_{H^2}(\rho, t) = 2^{1/2}(4\pi t)^{-3/2}\,\mathrm{e}^{-t/4}\int_{\rho}^{\infty}\frac{s\,\mathrm{e}^{-s^2/4t}}{(\cosh s-\cosh\rho)^{1/2}}\,\mathrm{d}s \tag{4.14}$$

is the heat kernel at the origin in hyperbolic space. Therefore the Bargmann representation is specified by the scalar product

$$\langle\phi|\psi\rangle = \int_{\boldsymbol{z}^2=1}\mathrm{d}\mu(\boldsymbol{z})(\phi(\boldsymbol{z}^*))^*\psi(\boldsymbol{z}^*). \tag{4.15}$$

The action of the operators J_i and X_j, $i, j = 12, 3$ in the Bargmann representation is given by

$$\boldsymbol{J}\phi(\boldsymbol{z}^*) = -\mathrm{i}\left(\boldsymbol{z}^*\times\frac{\partial}{\partial\boldsymbol{z}^*}\right)\phi(\boldsymbol{z}^*) \tag{4.16a}$$

$$X_1\phi(\boldsymbol{z}^*) = -\frac{\mathrm{i}}{\sinh 1}\left(\mathrm{e}^{J_2-\boldsymbol{J}^2/2}z_3^*\,\mathrm{e}^{\boldsymbol{J}^2/2-J_2} - \cosh 1\,\mathrm{e}^{-\boldsymbol{J}^2/2}z_3^*\,\mathrm{e}^{\boldsymbol{J}^2/2}\right)\phi(\boldsymbol{z}^*) \tag{4.16b}$$

$$X_2\phi(\boldsymbol{z}^*) = \frac{\mathrm{i}}{\sinh 1}\left(\mathrm{e}^{J_1-\boldsymbol{J}^2/2}z_3^*\,\mathrm{e}^{\boldsymbol{J}^2/2-J_1} - \cosh 1\,\mathrm{e}^{-\boldsymbol{J}^2/2}z_3^*\,\mathrm{e}^{\boldsymbol{J}^2/2}\right)\phi(\boldsymbol{z}^*) \tag{4.16c}$$

$$X_3\phi(\boldsymbol{z}^*) = \mathrm{e}^{-\boldsymbol{J}^2/2}z_3^*\,\mathrm{e}^{\boldsymbol{J}^2/2}\phi(\boldsymbol{z}^*). \tag{4.16d}$$

As remarked by Hall [19] the Bargmann representation (4.15) is a concrete realization of the general mathematical scheme of construction of Bargmann spaces introduced in [20].

As with the coherent states for a circle and torus discussed in previous sections our criterion of closeness of the coherent states for the quantum mechanics on a sphere to points of the classical phase space is behaviour of expectation values of the angular momentum and the position of a particle. From computer calculations it follows that

$$\frac{\langle \boldsymbol{l}, \boldsymbol{x}|\boldsymbol{J}|\boldsymbol{l}, \boldsymbol{x}\rangle}{\langle \boldsymbol{l}, \boldsymbol{x}|\boldsymbol{l}, \boldsymbol{x}\rangle} \approx \boldsymbol{l} \tag{4.17}$$

where $|\boldsymbol{l}, \boldsymbol{x}\rangle \equiv |\boldsymbol{z}\rangle$, with $\boldsymbol{z}$ given by (4.7), and the approximation is a bit worse than in the case of the circle and torus. Namely, for $|\boldsymbol{l}| \geqslant 10$, the relative error is of order 1%. Thus $\boldsymbol{l}$ parametrizing the coherent states can be identified with the classical angular momentum. This interpretation of $\boldsymbol{l}$ is also confirmed by the behaviour of the distribution of energies of the rotator in the coherent state [16] given by

$$p_{j,m}(\boldsymbol{x}, \boldsymbol{l}) = \frac{|\langle j, m; r|\boldsymbol{x}, \boldsymbol{l}\rangle|^2}{\langle \boldsymbol{x}, \boldsymbol{l}|\boldsymbol{x}, \boldsymbol{l}\rangle} \qquad -j \leqslant m \leqslant j. \tag{4.18}$$

Namely, for fixed integer $m = l_3$ the function $p_{j,m}$ has a maximum at $j_{\max}$ coinciding with the integer nearest to the positive root of the equation

$$j(j+1) = \boldsymbol{l}^2. \tag{4.19}$$

Therefore $\boldsymbol{l}^2/2$ can be regarded as the classical energy of the particle. Furthermore, for fixed integer j in $p_{j,m}$ satisfying (4.19), the function $p_{j,m}$ is peaked at $m_{\max}$ coinciding with the integer nearest to l_3. This means that the parameter l_3 can really be identified with the third component of the classical momentum. Furthermore, the computer simulations indicate that

$$\frac{\langle \boldsymbol{l}, \boldsymbol{x}|\boldsymbol{X}|\boldsymbol{l}, \boldsymbol{x}\rangle}{\langle \boldsymbol{l}, \boldsymbol{x}|\boldsymbol{l}, \boldsymbol{x}\rangle} \approx \mathrm{e}^{-1/4}\boldsymbol{x}, \tag{4.20}$$

where the accuracy of the approximation (4.20) is the same as for (4.17). Proceeding analogously as in the case of the circle and torus we can introduce relative expectation values of X_i with respect to averages in the coherent states labelled by the unit vectors $\boldsymbol{x} = \boldsymbol{e}_i$, $i = 1, 2, 3$ in (4.7) [16], so that

$$\langle\!\langle \boldsymbol{X} \rangle\!\rangle_{(\boldsymbol{l},\boldsymbol{x})} \approx \boldsymbol{x} \tag{4.21}$$

therefore, the parameter $\boldsymbol{x}$ can be regarded as the classical position on a sphere.

We now identify the wavefunctions corresponding to the coherent states. The coordinate representation for the quantum mechanics on a sphere is spanned by the common eigenvectors $|\boldsymbol{x}\rangle$ of the position operators X_i, that is

$$\boldsymbol{X}|\boldsymbol{x}\rangle = \boldsymbol{x}|\boldsymbol{x}\rangle \tag{4.22}$$

where we restrict, without loss of generality, to the irreducible representations with $r = 1$, so $\boldsymbol{X}^2 = 1$, and $\mathbf{x}^2 = 1$. The resolution of the identity is then given by

$$\int_{S^2} \mathrm{d}\nu(\boldsymbol{x})|\boldsymbol{x}\rangle\langle\boldsymbol{x}| = I \tag{4.23}$$

where $\mathrm{d}\nu(\boldsymbol{x}) = \sin\theta\,\mathrm{d}\varphi\,\mathrm{d}\theta$, where θ, φ are spherical coordinates, so $\boldsymbol{x} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$. The resolution of the identity (4.23) leads to the functional representation $L^2(S^2)$ of vectors such that

$$\langle\phi|\psi\rangle = \int_{S^2} \mathrm{d}\nu(\boldsymbol{x})\phi^*(\boldsymbol{x})\psi(\boldsymbol{x}), \tag{4.24}$$

where $\phi(\boldsymbol{x}) = \langle\boldsymbol{x}|\phi\rangle$. The operators $\boldsymbol{J}$ and $\boldsymbol{X}$ act in the representation (4.24) in the following way:

$$\boldsymbol{J}\phi(\boldsymbol{x}) = -\mathrm{i}\left(\boldsymbol{x} \times \frac{\partial}{\partial \boldsymbol{x}}\right)\phi(\boldsymbol{x}) \qquad \boldsymbol{X}\phi(\boldsymbol{x}) = \boldsymbol{x}\phi(\boldsymbol{x}), \tag{4.25}$$

where $\partial/\partial \boldsymbol{x} = (\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3)$ is the gradient operator. Using the remarkable identity

$$\boldsymbol{Z} = \mathrm{e}^{-\boldsymbol{J}^2/2} \boldsymbol{X}\, \mathrm{e}^{\boldsymbol{J}^2/2} \tag{4.26}$$

we find that the wavefunctions corresponding to the coherent states can be written as [18]

$$\phi_{(\boldsymbol{l},\boldsymbol{y})}(\boldsymbol{x}) = \frac{1}{\sqrt{4\pi}} \sum_{j=0}^{\infty} \mathrm{e}^{-(1/2)j(j+1)}(2j+1)P_j(\boldsymbol{x}\cdot\boldsymbol{z}) \tag{4.27}$$

where $\phi_{(\boldsymbol{l},\boldsymbol{y})}(\boldsymbol{x}) = \langle \boldsymbol{x}|\boldsymbol{l},\boldsymbol{y}\rangle$, and $\boldsymbol{z} = \cosh|\boldsymbol{l}|\frac{\boldsymbol{y}}{r} + \mathrm{i}\frac{\sinh|\boldsymbol{l}|}{|\boldsymbol{l}|}\boldsymbol{l}\times\frac{\boldsymbol{y}}{r}$, where $\boldsymbol{y}^2 = r^2$ and $\boldsymbol{l}\cdot\boldsymbol{y} = 0$. The computer simulations show that the probability density $p_{(\boldsymbol{l},\boldsymbol{y})}(\boldsymbol{x})$ for the coordinates in the normalized coherent state given by

$$p_{(\boldsymbol{l},\boldsymbol{y})}(\boldsymbol{x}) = \frac{|\phi_{(\boldsymbol{l},\boldsymbol{y})}(\boldsymbol{x})|^2}{\|\phi_{(\boldsymbol{l},\boldsymbol{y})}\|^2} = \frac{1}{4\pi}\frac{\left|\sum_{j=0}^{\infty}\mathrm{e}^{-(1/2)j(j+1)}(2j+1)P_j(\boldsymbol{x}\cdot\boldsymbol{z})\right|^2}{\sum_{j=0}^{\infty}\mathrm{e}^{-j(j+1)}(2j+1)P_j(|\boldsymbol{z}|^2)} \tag{4.28}$$

where $|\boldsymbol{z}|^2 = \sum_{i=1}^{3}|z_i|^2$, is peaked at $\boldsymbol{x} = \boldsymbol{y}$. This result supports the interpretation of the parameter $\boldsymbol{x}$ in (4.7) as the classical position of a particle on a sphere.

We end this section with some general remarks. An interesting problem is to generalize the construction of the coherent states for the quantum mechanics on a sphere S^2 to the case of $\lambda \neq 0$ (see (4.2)). A hint is the identification of classical variables parametrizing the cotangent bundle referring to the case $\lambda \neq 0$ [21] suggesting the following generalization $\boldsymbol{L}$ of the operator $\boldsymbol{J}$ in formula (4.8):

$$\boldsymbol{L} = \boldsymbol{J} - \frac{\lambda}{r^2}\boldsymbol{X} \tag{4.29}$$

where $\boldsymbol{L}\cdot\boldsymbol{X} = 0$. Unfortunately, we did not succeed in using (4.29) for construction of the coherent states. The technical reason was the problem with commutativity of the components of the operator $\boldsymbol{Z}$. We also point out that the coherent states for a circle S^1 and sphere S^2 were generalized to the case of an n-dimensional sphere S^n by Hall [22]. The range of possible applications of the introduced coherent states seems to be exceptionally wide. In fact, the quantum mechanics on a sphere is one of the most extensively studied quantum systems discussed in many textbooks. Interestingly, we have found citations of our papers on coherent states for a particle on a sphere in a geophysical journal [23].

5. Conclusion

In this work we have presented our recent results concerning coherent states for a particle on such compact manifolds as a circle, torus and sphere. We point out that all coherent states discussed herein are eigenvectors of some non-Hermitean operators possessing polar decomposition such that the Hermitean part is a function of the momentum and a unitary part is connected with the position of a quantum particle (see (2.8), (3.5) and (4.8)). It is suggested that such a polar decomposition takes place in a general case of an arbitrary compact configuration manifold. Open problems connected with the coherent states include uncertainty relations. In the case of the coherent states for a circle the introduced uncertainty relations (2.23) need further studies [8, 9]. On the other hand, the authors do not know uncertainty relations minimized by the coherent states for a sphere. In spite of difficulties, an importance of models involving compact configuration space indicate that the presented coherent states would be a useful tool in quantum physics.

Acknowledgments

We would like to thank a referee for helpful comments. This work was partially supported by a University of Lodz grant.

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